Velocity Profile for Fully Developed Turbulent Flow in a Pipe

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An equivalent Poiseuille's Law is derived for a homogeneous isotropic turbulent field. The derivation is based on an analogy between momentum transfer and heat and mass transfer, three coefficients being used to characterize the exchange process: (1) the molecular viscosity μ, (2) an intensity parameter to characterize the magnitude of the turbulent velocity fluctuations $\overline{v^2}$, and (3) a scale parameter to characterize the scale of the turbulence τ.

Poiseulle's Law describes the velocity profile or the pressure gradient of a fluid in laminar flow in a pipe far from the entry. The law relates the velocity field to the pressure gradient through a transfer coefficient, the molecular viscosity. For a Newtonian fluid the viscosity is a property of the fluid alone and not of the flow.

In a pipe flow the fluid exerts a drag or shear stress at the wall; since force is the time rate of change of momentum, this shear stress constitutes a flux of momentum to the wall with units of momentum per unit time per unit area. The velocity gradient in the fluid constitutes a gradient of momentum ρU, with units of momentum per unit volume. Fully developed laminar flow in a pipe may be viewed as a transfer process in which momentum flows to the wall as a result of the momentum gradient in the fluid. The kinetic theory of gases pictures the transfer process as occurring from the mixing of fluid elements of different velocities through the random motion of the molecules.

Fully developed turbulent flow in a pipe can also be looked upon as a transfer process in the manner described above. However, the mixing of fluid elements occurs by the random convective motions of the turbulence as well as by the molecular movements; the transfer process should be related to the properties of the turbulence as well as molecular properties. A theoretical description of turbulent transport will depend on the definition of the parameters which characterize the turbulence, just as the molecular velocity and molecular mean-free-path characterize molecular transport. In recent years considerable progress has been made in the description of turbulence. However, only limited success has been experienced in applying these descriptions to the problem of turbulent transport. Velocity-profile and pressure-drop data for fully developed turbulent flow in a pipe have not been predicted in terms of parameters describing the turbulence. Instead the data have been correlated empirically.

Three notable contributions which attempt to relate turbulent transport to properties of the turbulence are the concept of the Reynolds stress; the mixing-length theories of Prandtl, Karman, and Taylor; and the statistical description of point source diffusion by Taylor.

Reynolds showed that the shear stress in turbulent pipe flow may be described by the following equation:

$$S = -\rho\overline{uv} \tag{1}$$

The difficulty in applying this to a description of the velocity profile is that one cannot predict how $\overline{uv}$ varies throughout the flow field.

The mixing-length theories depend on the definition of a mixing length l and a turbulent intensity $\overline{v^2}$, analogous to the mean-free-path and molecular velocity defined for molecular motion. The application of these theories depends upon the relation of the mixing length to the velocity gradient.

Taylor described the turbulent mixing from a point source of material in terms of the turbulent intensity $\overline{v^2}$ and a time scale τ:

$$\tau = \int_0^\infty \frac{\overline{v_t v_{t'}}}{\overline{v^2}}\, dt \tag{2}$$

Taylor's theory is the most successful attempt at a description of the turbulent mixing process. It suffers from the limitation that it was derived for a turbulence which is homogeneous, while most actual turbulent fields are not homogeneous. Nevertheless, it supplies a basis for looking at actual situations where turbulent transport is occurring and for examining the effect of the nonhomogeneities on the transport process. One of the authors (*5*) has applied the Taylor description of turbulent mixing to the case of heat transfer from a hot plane wall to a cold plane wall through a turbulent air stream. The calculated temperature profiles approximated experimental measurements. It was shown that the increase in the resistance to heat transfer in the vicinity of the wall could be explained to a large degree, by the time dependency of the diffusion process rather than by a change in the flow characteristics. This time dependency arises since the distance scale of the turbulence is of the same magnitude as the size of the container. An increased resistance to heat transfer in this case arises since molecules in the vicinity of the wall that have exchanged heat with the wall are in the field a shorter length of time than those in the center of the channel.

If it is assumed that momentum diffuses in the same manner as heat and mass in a turbulent field, then Taylor's theory of point-source diffusion can be applied to the description of the velocity field for fully developed pipe flow. The velocity field will be calculated as a function of the shear stress at the wall or the pressure gradient. The parameters entering into the calculation are the molecular viscosity μ, the turbulence intensity $\overline{v^2}$, and the time scale τ.

Since the turbulence in pipe flow is nonhomogeneous the description presented in this paper might be considered a first approximation to a Poiseuille's Law for turbulent flow.

CONTRIBUTION OF TURBULENCE TO THE TRANSPORT PROCESS

Taylor (*7*) in 1935 described the spread of material from a point source in terms of the variation of mean square displacement of the particles $\overline{X^2}$ with time. He showed that for a homogeneous isotropic field

$$\frac{d\overline{X^2}}{dt} = 2\overline{v^2}\int_{t'}^{t} R_{t-t'}\, dt \tag{3}$$

Experiments have indicated that the Lagrangian correlation coefficient, $R_{t-t'}$, can be approximated by

$$R_{t-t'} = \exp\left\{\frac{-(t-t')}{\tau}\right\} \tag{4}$$

where

$$\tau = \int_{t'}^{\infty} R_{t-t'}\, dt \qquad (5)$$

Equation (3) becomes

$$\frac{d\overline{X^2}}{dt} = 2\overline{v^2}\tau \left[1 - \exp\left\{-\frac{(t - t')}{\tau}\right\}\right] \qquad (6)$$

For molecular diffusion the rate of change of $\overline{X^2}$ with time is constant,

$$\frac{d\overline{X^2}}{dt} = 2D \qquad (7)$$

Turbulent diffusion however is time dependent in that $d\overline{X^2}/dt$ is a function of time. The parameters $\overline{v^2}$ and τ may be obtained from measurements of the concentration profiles at different distances downstream from a point source of material in a flow stream (*4*), which may consist of a small tube from which a foreign substance is injected into the stream. From these data the variation of $\overline{X^2}$ with time is obtained, and $\overline{v^2}$ and τ can be calculated from Equation (6).

Measurements have shown that the distribution function describing $\overline{X^2}$ may be approximated by the Gaussian error curve. For such a case it can be shown (*5, 1*) that diffusion from an infinitesimal source can be described by the differential equation:

$$\frac{\partial C}{dt} = \beta\, \nabla^2\, C \qquad (8)$$

where

$$\beta = \frac{1}{2}\frac{d\overline{X^2}}{dt} \qquad (9)$$

The use of Equation (8) allows statistical information on the variation of $\overline{X^2}$ with time to be translated into information on the concentration distribution. It is also more convenient than a purely statistical approach for treating problems where the infinitesimal source or sink is not a point. In this paper a sink will be considered which consists of an infinitesimal ring around the inside diameter of a pipe.

Equation (8) is valid only for the description of the behavior of an infinitesimal source or sink. However, an actual case involving turbulent transport can be described in terms of a distribution of sources or sinks. Thus heat transfer from a hot to a cold wall through a turbulent fluid can be viewed as consisting of a series of heat sources along the hot wall and a series of heat sinks along the cold wall. The temperature field can be calculated as the sum of the contributions from all these sources and sinks (*5*).

CONTRIBUTION OF MOLECULAR TRANSPORT

The derivation of Taylor applied only to turbulent diffusion and did not consider the role of molecular diffusion in the over-all transport process. Molecular diffusion can contribute to the transport process in two ways: (1) It can act as a mechanism for diffusion in addition to the turbulent transport. This will increase the total transport rate. (2) It can decrease the effective scale τ for turbulent diffusion. This arises since material which is moving in an eddy will tend to escape from the eddy because of molecular diffusion. No experimental data are available to indicate the magnitude of this effect. For the case considered in this paper it is probably not too great. It will be assumed that the effect of molecular diffusion upon turbulent transport can be approximated by altering the value of τ appearing in Equation (6). The total transport due to molecular and turbulent diffusion can then be represented by the following equation:

$$\frac{d\overline{X^2}}{dt} = 2D + 2\overline{v^2}\tau_c \left[1 - \exp\left\{-\frac{(t - t')}{\tau_c}\right\}\right] \qquad (10)$$

τ_c = time scale corrected for the effect of molecular diffusion.

MODEL FOR DESCRIBING FULLY DEVELOPED PIPE FLOW

A description of the velocity field occurring in fully developed pipe flow will be carried out by assuming that momentum diffuses in a manner analogous to the diffusion of mass and heat. The statement of the analogy is the assumption that the behavior of an infinitesimal source or sink of momentum can be described by the following equation:

$$\frac{\partial(\rho U)}{\partial t} = \beta\, \nabla^2\, (\rho U) \qquad (11)$$

where

$$\beta = \frac{1}{2}\frac{d\overline{X^2}}{dt} = \frac{\mu}{\rho} + \overline{v^2}\tau_c \left[1 - \exp\left\{-\frac{(t - t')}{\tau_c}\right\}\right] \qquad (12)$$

In order to apply this analogy to the calculation of the velocity field for fully developed turbulent flow the model outlined in the introduction to this paper will be used to describe the momentum transfer process. The validity of the model is independent of the mechanism assumed for the transfer process. Its accuracy will depend on the accuracy with which the transfer process can be

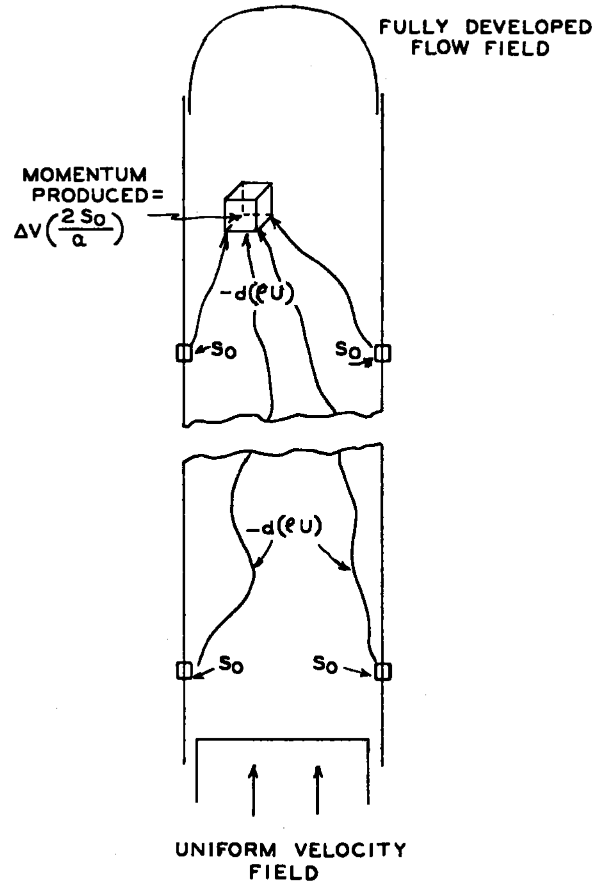

Fig. 1. Model to express analogy for fully developed pipe flow.

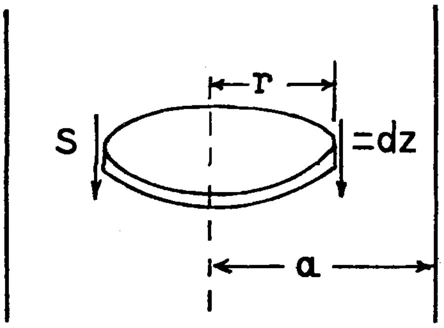

Fig. 2. Differential volume in flow field.

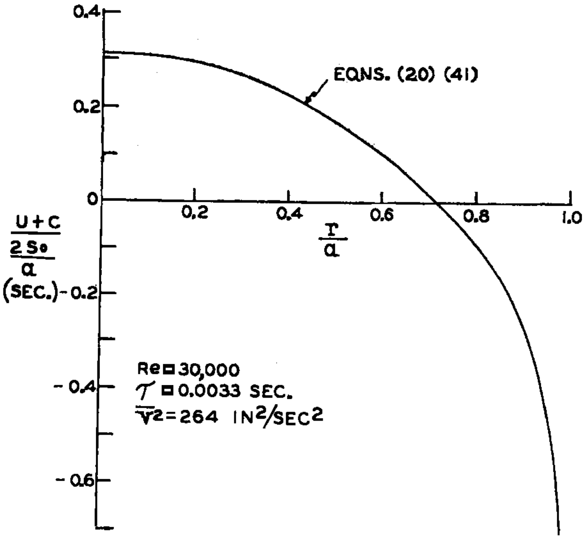

Fig. 3. Time-dependent turbulent diffusion.

described. Therefore, for laminar flow the model should result in a derivation of Poiseuille's Law.

Figure 1 depicts the phenomenon that is occurring. Since the final flow field is independent of the entry conditions, any convenient entry flow profile may be assumed as long as momentum is conserved. The fluid is assumed to have a uniform flow at the pipe entry. The pressure gradient is assumed constant. The exchange of momentum with the wall may be calculated by writing force balance over an increment of pipe length. It is a constant and is given by the equation

$$S_0 = \frac{a}{2}\frac{\Delta p}{\Delta z} \tag{13}$$

Owing to this exchange of momentum with the wall with the wall momentum (ρU) gradients are set up in the fluid. The initial velocity changes after the fluid enters the pipe. Experiments show that far enough along the pipe the shape of the velocity profile ceases to change. A steady state is reached in which there is no further change in the flow field and yet there is a constant exchange of momentum with the wall. In order for this to be possible there must be a creation of momentum in the fluid equal to that transferred to the wall. By writing a momentum balance around an element of fluid (Figure 2) one can show that, if the shear stress is constant at the wall, the momentum produced per unit volume is constant.

$$\text{Momentum created} = \begin{matrix}\text{momentum}\\ \text{transferred}\end{matrix}$$

$$= 2\pi r S\, dz$$

$$\frac{\text{momentum created}}{\text{unit volume}} = \frac{2\pi r S\, dz}{\pi r^2\, dz}$$

$$= \frac{2S}{r}$$

For fully developed pipe flow,

$$\frac{S}{r} = \frac{S_0}{a} = \text{constant}$$

In this development, therefore, it is assumed that

$$\frac{\text{momentum created}}{\text{unit volume}} = \frac{2S_0}{a} \tag{14}$$

The model then consists of fluid flowing into a pipe with a uniform velocity. Upon flowing past the wall it exchanges momentum with the wall and velocity gradients are set up in the fluid. The wall constitutes a sink of momentum or velocity; a negative amount of momentum $-\rho U$ may be visualized as diffusing away from the wall. There is a production of momentum within the fluid equal to that transferred to the wall. A small element of fluid within the field may be considered as a region in which momentum is created and into which "negative" amounts of momentum are diffusing from sinks along the wall. Eventually a steady state is realized and the velocity profile undergoes no further change. A fully developed flow is attained. The model as presented contains no assumptions regarding the mode of transfer. Four examples will be explored: (1) molecular transfer (2) molecular transfer + turbulent transfer with no time dependency (3) turbulent transfer with time dependency, (4) molecular transfer + turbulent transfer with time dependency.

In the consideration of these different modes of transfer the problem will be reduced to one of considering a two-dimensional flow field of circular cross section. There will be a sink of momentum around the periphery and a continuous production within the fluid. The distribution of momentum within the field will be calculated as a function of time.

The behavior of an instantaneous infinitesimal source of negative momentum at the periphery is described as a solution to Equation (11) for the boundary conditions:

$$\begin{array}{lll}\text{at} & t = 0 & \rho U = 0\\ \text{at} & t = t' & \text{instantaneous sink of}\\ & r = a & \text{momentum} = -S_0\, dt'\\ \text{at} & t > t' & \dfrac{\partial(\rho U)}{\partial r} = 0\\ & r = a & \end{array}$$

The solution is similar to that given by Carslaw and Jaeger (*2*) for a constant diffusivity:

$$(\rho U)_{\substack{\text{single}\\ \text{source}}} = -\frac{2S_0\, dt'}{a} \sum_{s=0}^{\infty} e^{-\alpha_s{}^2 K(t-t')} \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)} \tag{15}$$

$$K(t - t') = \int_{t'}^{t} \beta\, dt$$

$$= \int_{t'}^{t} \frac{1}{2}\frac{d\overline{X^2}}{dt}\, dt \tag{16}$$

The quantity α_s is the root of $J_0'(\alpha_s a) = 0$. Equation (15) describes the following situation. At time t' a finite amount of negative momentum is emitted into the field at $r = a$. At all other times there is no transfer of momentum at $r = a$. Momentum diffuses about the field without escaping from it.

Since there is a continuous diffusion of momentum from the wall, the contribution of all sinks from time = 0 to time = t would be the integral of the foregoing expression.

$$\rho U = -\int_0^t \frac{2S_0}{a} \sum_{s=0}^{\infty} e^{-\alpha_s{}^2 K(t-t')} \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\, dt' \tag{17}$$

The momentum created between r and $r + dr$ during this interval is

$$\rho U = \int_0^t \frac{2S_0}{a}\, dt' \tag{18}$$

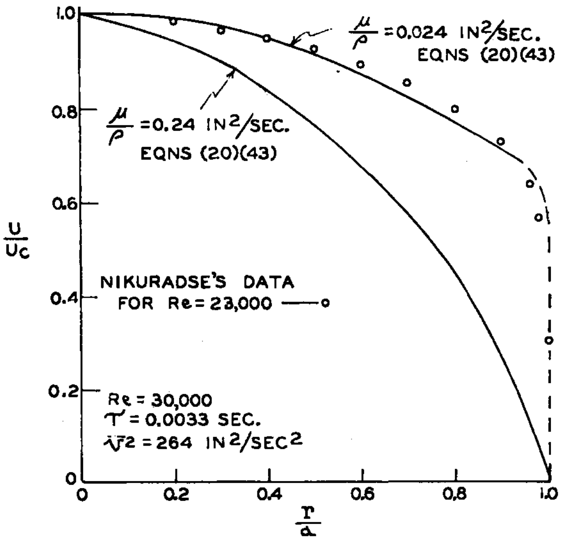

Fig. 4. Molecular diffusion by time-dependent turbulent diffusion.

The velocity field may be represented by the sum of Equations (17) and (18) except for an additive constant.

$$\rho U + C_1 = \int_0^t \frac{2S_0}{a}\,dt' - \int_0^t \frac{2S_0}{a} \sum_{s=0}^{\infty} e^{-\alpha_s^2 K(t-t')} \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\,dt' \qquad (19)$$

For the condition of fully developed flow the integral is evaluated for $t = \infty$. Since for $s = 0$, $\alpha_s = 0$,

$$U \times C_2 = -\int_0^{\infty} \frac{2S_0}{\rho a} \sum_{s=1}^{\infty} e^{-\alpha_s^2 K(t-t')} \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\,dt' \qquad (20)$$

Since at $r = a$, $U = 0$,

$$C_2 = -\frac{2S_0}{\rho a}\int_0^{\infty} \sum_{s=1}^{\infty} e^{-\alpha_s^2 K(t-t')}\,dt' \qquad (21)$$

The function $K(t - t')$ will depend on the description of the transfer process. The four possibilities that have been outlined will now be considered.

CASE I: MOLECULAR EXCHANGE

If the transfer of momentum occurs only by molecular exchange, the quantity β is constant and equal to the kinematic viscosity, μ/ρ:

$$K(t - t') = \frac{\mu}{\rho}(t - t') \qquad (22)$$

If this is substituted into Equation (20) and if the integration is carried out, the following expression results for the velocity field:

$$U + C_2 = -\frac{2S_0}{a} \sum_{s=1}^{\infty} \frac{1}{\mu \alpha_s^2} \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)} \qquad (23)$$

$$C_2 = -\frac{2S_0}{a} \sum_{s=1}^{\infty} \frac{1}{\mu \alpha_s^2} \qquad (24)$$

Equation (23) describes a parabola. This can be demonstrated as follows: Since

$$U_c = \frac{2S_0}{\mu a} \sum_{s=1}^{\infty} \frac{1}{\alpha_s^2}\left[1 - \frac{1}{J_0(\alpha_s a)}\right] \qquad (25)$$

Equation (23) can be expressed in dimensionless form as

$$\frac{U}{U_c} = \frac{\sum_{s=1}^{\infty} \frac{1}{\alpha_s^2}\left[1 - \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\right]}{\sum_{s=1}^{\infty} \frac{1}{\alpha_s^2}\left[1 - \frac{1}{J_0(\alpha_s a)}\right]} \qquad (26)$$

The equation for a parabola is

$$\frac{U}{U_c} = 1 - \left(\frac{r}{a}\right)^2 \qquad (27)$$

To show that these two expressions are equal one may expand Equation (27) as a series of Bessel functions as described in Wylie (*8*).

$$1 - \left(\frac{r}{a}\right)^2 = A_0 + \sum_{s=1}^{\infty} A_s J_0(\alpha_s r) \qquad (28)$$

As defined previously, α_s is a root of the equation

$$J_0'(\alpha_s a) = -J_1(\alpha_s a) = 0 \qquad (29)$$

The constants A_s can be evaluated from the orthogonality property

$$\int_0^a r J_0(\alpha_s r) J_0(\alpha_n r)\,dr = \frac{a^2}{2} J_0^2(\alpha_s a) \quad n = s$$
$$= 0 \quad n \neq s \qquad (30)$$

Both sides of Equation (28) are multiplied by $rJ_0(\alpha_s r)\,dr$ and integrated. Since

$$\int_0^a r J_0(\alpha_s r)\,dr = 0 \qquad (31)$$

$$\int_0^a \frac{r^3}{a^2} J_0(\alpha_s r)\,dr = \frac{2}{\alpha_s^2} J_0(\alpha_s a) \qquad (32)$$

the constant A_s becomes

$$A_s = -\frac{4}{\alpha_s^2 a^4 J_0(\alpha_s a)} \qquad (33)$$

and the series representation of $1 - (r/a)^2$ is

$$1 - \left(\frac{r}{a}\right)^2 = A_0 - \sum_{s=1}^{\infty} \frac{4 J_0(\alpha_s r)}{\alpha_s^2 a^2 J_0(\alpha_s a)} \qquad (34)$$

At $r = a$

$$0 = A_0 - \sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2} \qquad (35)$$

$$A_0 = \sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2} \qquad (36)$$

At $r = 0$

$$1 = A_0 - \sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2 J_0(\alpha_s a)} \qquad (37)$$

If Equation (37) is divided into Equation (34) and if Equation (36) is substituted for A_0,

$$1 - \left(\frac{r}{a}\right)^2 = \frac{\sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2} - \sum_{s=1}^{\infty} \frac{4 J_0(\alpha_s r)}{\alpha_s^2 a^2 J_0(\alpha_s a)}}{\sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2} - \sum_{s=1}^{\infty} \frac{4}{\alpha_s^2 a^2 J_0(\alpha_s a)}} = \frac{\sum_{s=1}^{\infty} \frac{1}{\alpha_s^2}\left[1 - \frac{J_0(\alpha_s r)}{J_0(\alpha_s a)}\right]}{\sum_{s=1}^{\infty} \frac{1}{\alpha_s^2}\left[1 - \frac{1}{J_0(\alpha_s a)}\right]} \qquad (38)$$

The right-hand side of Equation (38) is identical with the right-hand side of Equation (26). The expression derived on the basis of the model used to describe fully developed pipe flow yields the same result as the integration of the Navier Stokes equation for the case where momentum is transported only by molecular exchange. The model is a valid one. Other modes of momentum transport may now be investigated with it.

CASE II: MOLECULAR EXCHANGE + TIME-INDEPENDENT TURBULENT EXCHANGE

Quite often the participation of turbulence in the transfer process is taken into account by the assumption that the transfer coefficient is described as the sum of a molecular diffusivity and eddy diffusivity, ϵ.

$$\beta = \frac{\mu}{\rho} + \epsilon \qquad (39)$$

This is not an accurate representation of turbulent diffusion since it does not account for the length of time the diffusing material has been in the field. It is an empirical approach to the problem; the eddy diffusivity, in general, will vary in an unknown manner throughout the field. The effect of the presence of turbulence is to increase β markedly over that for molecular exchange only. If ϵ were constant, the velocity field is described by Equation (23) with $\mu + \rho\epsilon$ substituted for μ. The profile again would be a parabola. However, for a given pressure gradient or S_0/a the magnitude of the velocities would be much smaller.

CASE III: TIME-DEPENDENT TURBULENT EXCHANGE

The method used by Taylor to describe the time dependency of turbulent diffusion has been outlined in the beginning of the article. If the turbulent-exchange process is described in this manner and if the contribution of molecular diffusion is neglected,

$$\beta = \overline{v^2}\tau\left[1 - \exp\left\{-\left(\frac{t-t'}{\tau}\right)\right\}\right] \qquad (40)$$

$$K(t - t') = \overline{v^2}\tau(t - t') - \overline{v^2}\tau^2 + \overline{v^2}\tau^2 \exp\left\{-\left(\frac{t-t'}{\tau}\right)\right\} \qquad (41)$$

This may be substituted into Equation (20) to evaluate $U + C_2$. The resulting expression is plotted in Figure (3) by use of values of τ and $\overline{v^2}$ that were obtained from point-source diffusion measurements. It goes to minus infinity at $r = a$, therefore it is not possible to evaluate the constant of integration. In the immediate vicinity of the wall the resistance to the transport of material away from the wall by turbulent diffusion is very large. Right at the wall the rate of transfer is zero. This can be seen by examining Equation (40). For $(t - t') = 0$, $\beta = 0$. In the vicinity of the wall molecular exchange becomes important. It cannot be neglected.

CASE IV: TIME-DEPENDENT TURBULENT EXCHANGE + MOLECULAR EXCHANGE

If the contribution of molecular exchange is considered along with a time-dependent turbulent diffusion, β and $K(t - t')$ are given by the following two equations:

$$\beta = \frac{\mu}{\rho} + \overline{v^2}\tau_c \cdot \left[1 - \exp\left\{-\left(\frac{t-t'}{\tau_c}\right)\right\}\right] \qquad (42)$$

$$K(t - t') = \overline{v^2}\tau_c(t - t') - \overline{v^2}\tau_c^2 + \overline{v^2}\tau_c^2 \exp\left\{-\left(\frac{t-t'}{\tau_c}\right)\right\} + \frac{\mu}{\rho}(t - t') \qquad (43)$$

It is seen that at the wall β does not vanish. The incorporation of the contribution of molecular exchange gets around the difficulty of a velocity of minus infinity at the wall. Equation (20) is plotted in Figure 4 by use of the foregoing expression for $K(t - t')$. Values of $\overline{v^2}$ and τ_c used in the calculations were taken from measurements of turbulent diffusion of hydrogen in air at a Reynolds number of 30,000. The measurements were not too accurate; no attempt was made to correct them for molecular exchange. Recent measurements in this laboratory have indicated values of τ and $\overline{v^2}$ of 0.0044 sec. and 200 sq. in./sec.^{2} Velocity measurements of Nikuradse (*6*) are plotted on the same graph. The agreement is within the errors involved in the selection of $\overline{v^2}$ and τ_c.

On the same graph a calculated velocity profile using a value of μ/ρ tenfold greater than that of air is presented. It can be seen that Equation (20) is sensitive to the value of μ/ρ used. However, as pointed out in an earlier part of this paper, the curve for $\mu/\rho = 0.24$ sq. in./sec. may not represent the effect of a change of the molecular viscosity upon the velocity profile. This arises from the fact that the scale represented the turbulent exchange will decrease with an increase in the molecular transport properties and from the fact that the Reynolds number will change.

CONCLUSIONS

The model used to establish an analogy between momentum transfer and heat and mass transfer is valid. This was shown by the fact that the velocity profile calculated for laminar flow agreed with Poiseuille's Law. The mathematical formulation of the model for a turbulent field depends on the assumptions:

1. Diffusion from an infinitesimal source can be described by Equation (11).
2. There is an analogy between momentum transfer and heat and mass transfer in that β for momentum transfer can be obtained from point-source diffusion measurements.
3. The role of turbulence in effecting the transport of momentum may be described by employing a homogenous field.

The expression for the velocity profile depends upon the description of transfer process. A transport of momentum by means of time-dependent turbulent diffusion and molecular diffusion describes the velocity field within the approximations employed in the derivation of Equation (42) to represent the transport process and within the accuracy of the experimental parameters used in the calculation.

NOTATION

a = radius of the pipe
C_1, C_2 = constants
C = concentration
D = molecular diffusion coefficient
$J_0(x)$ = Bessel function of first kind of order zero
p = pressure
R = Lagrangian correlation coefficient = $\overline{v_t v_{t'}}/\overline{v^2}$
r = radial distance
Re = Reynolds number
S = shear stress at r
S_0 = shear stress at the wall
t = time
t' = time at which diffusible material entered the turbulent field
$\overline{uv}$ = time average of the product of the turbulent velocity component in the x direction and the z direction
U = time-average velocity in the z direction
U_c = center velocity
v_t = instantaneous velocity at time t of a particle in the x direction
v_t' = instantaneous velocity at time t' of a particle in the x direction
$\overline{v_t v_{t'}}$ = average of v_t, v_t' for a large number of particles
$\overline{v^2}$ = mean-square turbulent velocity in the x direction
ΔV = increment of volume
z = direction of average flow
α_s = root of $J_0'(\alpha_s a) = 0$
β = $\frac{1}{2}\overline{dX^2}/dt$
ϵ = eddy-diffusion coefficient
μ = fluid viscosity
ρ = fluid density
τ = Lagrangian time scale = $\int_0^\infty R\,dt'$
τ_c = Lagrangian time scale corrected for effects of molecular transport

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